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Plane Quintic Curves which Possess a Group of Linear Transformations.

BY VIRGIL SNYDER.

1. Plane curves which are invariant under linear transformations are important because they furnish examples of configurations of points of inflexion which disprove Grassmann's theorem.[*]

Thus far no general positive results have been obtained regarding the position and dependence of such points, and each new concrete case is of interest. The purpose of this paper is to discuss the plane quintic curves which are left invariant by linear transformations, without considering the important cases that are self-dual.

2. Finite ternary linear groups may be divided into three classes:

a) Those leaving a triangle invariant;
b) The regular body groups;
c) Ternary groups not belonging to a) nor b).

Of c), the only forms which have been found are G_{216} (Jordan), G_{168} (Klein), and G_{360} (Valentiner). The invariant form of G_{216} is the syzygetic pencil of c_3, that of the simple G_{168} is the c_4 defined by $x^3y + y^3z + z^3x = 0$, and its covariants of orders 6, 12, 21; that of the simple group G_{360} is a particular c_6 and its covariants of orders 12 30, 45.[†]

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[*] "Zur Theorie der Wendepunkte, besonders der Curven vierter Ordnung," by Justin Grassmann, Dissertation, Berlin, (1875). The theorem in question (p. 45) is: *If a c_{n-2} be passed through $\frac{1}{2}(n-2)(n+1)$ points of inflexion of a non-singular c_n, the remaining $\frac{1}{2}(n-1)(n-2)$ points of intersection will also be points of inflexion of c_n.* For $n=4$, compare the remark by Klein, *Math. Ann.* Vol. 10, p. 397, (1876).

[†] For the literature concerning these groups, see Encyclopädie, IB 2, vol. 1_1, pp. 339–340, and IB 3f, vol. 1_1, pp. 528–529. To this list three recent papers by Ciani may be added: "Un teorema sopra la quartica di Klein," *Rend. Ist. Lombardi* (2), vol. 33 (1900) pp. 565-566. "I gruppi finite di collineazioni . . . . ", ibid, pp. 1170–1175. "Contributo alla teoria del gruppo di 168 collineazioni piani." *Annali* (3), vol. 5, (1901) pp. 33-55.

Of the regular body groups, the dihedral group and the invariant G_4 of the tetrahedral group need not be considered, since they are contained in a). From the invariant forms of the remaining transformations it is seen that a c_5 cannot be transformed into itself, hence we need only consider groups of the form a). These may be either cyclical perspectivities, $x' = ax$, $y' = y$, $z' = z$, $a^k = 1$ or projections of rotations $x' = ax$, $y' = a^{-1}y$, $z' = z$, or combinations of the two. The order of the first cannot exceed 5, that of the second cannot exceed 10, but the third may have a higher order. In case of the cyclical perspective of order 5 the center cannot lie on the curve. The axis cuts the curve in points at each of which the tangent has five-point contact and passes through the center. No other tangents can be drawn from the center. If $k = 4$, the center is a simple point on c_5, at which the tangent has five-point contact. The axis cuts c_5 in five points at which the tangents have four-point contact and pass through the center. If $k = 3$, the center is a double point and each tangent has five-point contact. The tangents at the five points of inflexion on the axis pass through the center. Finally, if $k = 2$, the center is either a point of inflexion or a triple point, each branch having an inflexion at the point. In the former case, five tangents have their point of contact on the axis, and six bitangents pass through the center. In the latter case, only five tangents can be drawn from the center, and these have their points of contact on the axis.

It was shown by Maschke* that every curve which is invariant under a) is an integral function of xyz and $x^\alpha y^\beta + y^\alpha z^\beta + z^\alpha x^\beta = 0$, α, β being zero or positive integers. The invariant curves of the cyclic perspectivities will first be considered.

3. It can be easily shown that a c_5 cannot remain invariant under two harmonic homologies having a common axis. For, let $z = 0$ be the common axis, (0, 0, 1), (0, 1, 1) be the two centers.

The first homology is $\begin{pmatrix} x & y & z \\ x & y & -z \end{pmatrix}$, and the second is $\begin{pmatrix} x & y & z \\ x & y-2z & -z \end{pmatrix}$. If c_5 is invariant under the first, its equation can contain no odd powers of z,

$$z^4 f_1(x, y) + z^2 f_3(x, y) + f_5(x, y) = 0.$$

On making the second transformation and equating the coefficients of the new terms which appear to zero the equation is seen to be factorable.

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* "On ternary substitution groups of finite order which leave a triangle unchanged," JOURNAL, vol. 17 (1895), pp. 168-184.

Moreover, two concentric harmonic homologies cannot leave a c_5 invariant from the remark concerning the configuration of the tangents and bitangents from the center which we saw must lie on the curve. If A, A' be the centers and a, a' the axes of two harmonic homologies which leave c_5 invariant, either A lies on a', A' on a, or a third homology exists whose center is on AA'. In the first case, by taking a, a', AA' for axes, the transformations may be expressed by

$$\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}, \quad \begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix}.$$

This requires that x, y enter the equation in even powers only, which is impossible. Hence if c_5 admits two such homologies it also admits a third; if there be a third, such that its center is not on the line joining the first two, we must have at least nine, forming a sort of Hessian configuration. Each center is a point of inflexion on the curve. The line joining three centers cannot be a tangent to the curve, nor pass through a double point.

4. We now introduce the following notation:

$$U_1 \equiv \begin{pmatrix} x & y & z \\ x & z & y \end{pmatrix}, \quad T_1 \equiv \begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}, \quad U \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix},$$

$$S_1 \equiv \begin{pmatrix} x & y & z \\ \omega x & y & z \end{pmatrix}, \quad \omega^3 = 1; \qquad I_1 \equiv \begin{pmatrix} x & y & z \\ ix & y & z \end{pmatrix},$$

$$V_1 \equiv \begin{pmatrix} x & y & z \\ \theta x & y & z \end{pmatrix}, \quad \theta^5 = 1, \qquad U = U_1 U_2,$$

with corresponding symbols for similar changes in y and z. Let P_k represent a a k-fold point on the quintic curve c_5, $f_k(x, y)$ be a binary form in x, y of degree k, and $\phi_k(x, y)$ a symmetric form, so that $\phi_k(x, y) = \phi_k(y, x)$. Let G_m be a group of order m, in which $\begin{pmatrix} x & y & z \\ ax & ay & az \end{pmatrix}$ is regarded as identity.

5. The c_5 having P_4 at $(1, 0, 0)$ and symmetric in y, z

$$x\phi_4(y, z) + \phi_5(y, z) = 0$$

is invariant under U_1. P_4 lies on the axis $y - z = 0$. The center $(0, 1, -1)$ is a point of inflexion from which can be drawn two bitangents; the simple tangent from the center has its point of contact at the residual point on the axis. In particular, three forms may be considered that are non-symmetric:

$$\begin{aligned} axy^4 + by^5 + cz^5 &= 0, \\ axyz^3 + by^5 + cz^5 &= 0, \\ axy^4 + bz^5 &= 0. \end{aligned} \tag{1}$$

The first two have a cyclic G_5, generated by V_3, and the third a G_{20}, generated by I_2 and V_3. All the point and line singularities of this last curve are concentrated in the invariant points (0, 1, 0), (1, 0, 0).

The curve

$$axy^2 z^2 + b(y^5 + z^5) = 0 \tag{2}$$

has a G_{10}, generated by $V_1 V_2$ and U_1. It has five points of inflexion, all on $x = 0$, each one being the center of a harmonic homology. It has five bitangents, one through each point of inflexion. The curve

$$x(x^4 + y^4) + z(x^4 - y^4) = 0$$

is invariant under I_2. The tangent at (0, 1, 0) has five-point contact. The axis $y = 0$ cuts c_5 in (1, 0, — 1) and at P_4. In the former the tangent has four-point contact. The four remaining inflexions lie on a line passing through the center, upon which they form a harmonic range.

6. When c_5 has a P_3 and is invariant under T_1 its equation becomes

$$x^2 f_3(y, z) + f_5(y, z) = 0.$$

In particular, the curve

$$Az^2(x^3 + ay^3) + Bx^2(x^3 + by^3) = 0 \tag{3}$$

is invariant under T_3 and S_2. The center (0, 0, 1) of T_3 is a P_3 and (0, 1, 0), the center of S_2, is a P_2. Each tangent at both multiple points has five-point contact. The line $y = 0$ cuts c_5 in two points of inflexion; the remaining 12 are arranged in two hexads. The tangents at P_2 each count for one double tangent; the remaining 30 are grouped in five hexads.

The curve

$$ax^2 y^3 + bz^5 = 0 \tag{4}$$

is invariant under T_1, S_2, V_3, hence has a G_{30}. The two binomial curves are self-dual. They also have a continuous group.

7. The curve $x^3 f_2(y, z) + f_5(y, z) = 0$ has S_1, and

$$x^3 \phi_2(y, z) + \phi_5(y, z) = 0 \tag{5}$$

allows S_1 and U_1. The lines $f_5(y, z) = 0$ are all inflexional tangents passing through $P_2 \equiv (1, 0, 0)$, the points of inflexion lying on $x = 0$. The remaining 30 inflexions are arranged in groups of six, a line joining P_2 to any one will contain two others, and the line joining any one to (0, 1, — 1) will contain one other.

The curve

$$ax^3y^2+by^5+cz^5=0 \tag{6}$$

has a G_{15}, generated by S_1 and V_2. It has a cusp and adjacent double point at (1, 0, 0), the tangent $y=0$ having five-point contact. The side $z=0$ cuts c_5 in three points at which the tangents have five-point contact and pass through (0, 0, 1). The side $x=0$ cuts the curve in five ordinary inflexions, the tangents passing through the singular point. The latter absorbs two inflexions and two double tangents. The remaining 15 inflexions form a set, of which one is real. The 45 remaining double tangents are arranged in three such sets. If y^5 be replaced by y^4x, the curve has a G_{10} of type (2).

The curve

$$ax^3yz+b(y^5+z^5)=0 \tag{7}$$

is invariant under U_1, S_2 and $V_1 . V_3^2$, generating a G_{30}. The curve has a P_2 at (1, 0, 0), each tangent having five-point contact. $x=0$ cuts the curve in five ordinary inflexions and the remaining 30 are arranged in two sets, conjugate under U_1. Only two are real. The 30 points are also situated on 10 lines passing through the node. Each point of inflexion on $x=0$ is the center of a harmonic homology; six bitangents pass through each.

8. The curve $x^4f_1(y, z)+f_5(y, z)=0$ has the center (1, 0, 0) for a triple inflexion; four double inflexions lie on the axis of I_1 and the remaining 32 are arranged in eight harmonic groups on lines passing through the center.

The curve

$$ax^4y+b(y^5+z^5)=0 \tag{8}$$

is invariant under V_3 and I_1, making a G_{20}. The five points of c_5 on $x=0$, have four-point contact tangents, passing through (1, 0, 0) at which the tangent has five-point contact. Each point on $z=0$ has a five-point contact tangent. The remaining 20 form one cycle, two of them being real. They are arranged on four lines through (0, 0, 1), and on five through (1, 0, 0). No proper bitangent passes through the center of I_1.

9. The curve

$$x^5+\phi_5(y, z)=0 \tag{9}$$

belongs to V_1 and U_1. Five points of five-point contact tangents lie on $x=0$, and the remaining points of inflexion are arranged on six lines, any three of which compose an improper c_3, whose complete intersection with c_5 consists of points of inflexion, as does also the intersection of c_5 and $x^3 = 0$.

10. The curve

$$z^3 y^2 = x(x^4 + y^4) \tag{10}$$

has the G_{12} generated by $S_3 \,.\, T_3 \,.\, I_2$. It also has a Cremona group of order 72. It has a tacnode cusp at (0, 0, 1).

11. Of curves of the type $x^\alpha y^\beta + y^\alpha z^\beta + y^\alpha x^\beta = 0$, $\alpha, \beta \neq 0$ there are but two types to be considered.

The curve

$$x^4 z + z^4 y + y^4 x = 0 \tag{11}$$

is invariant under

$$U \equiv \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix} \text{ and } K \equiv \begin{pmatrix} x & y & z \\ x & \varepsilon y & \varepsilon^4 z \end{pmatrix}, \qquad \varepsilon^{13} = 1.$$

At each vertex of the triangle of reference a side has four point contact. U is the product of

$$K_1 \equiv \begin{pmatrix} x & y & z \\ x & \varepsilon y & \varepsilon^{-1} z \end{pmatrix} \text{ and } K_2 \equiv \begin{pmatrix} x & y & z \\ x & y & \varepsilon^5 z \end{pmatrix}.$$

The first is the projection of a rotation about (1, 0, 0), $y = 0$, $z = 0$ being the isotropic lines, and K_2 is a similarity transformation about (0, 0, 1), through which a real point goes into an imaginary point and return to its original positions after 12 operations.* Since reality is not changed by K_1 it follows that there can be at most three real points of inflexion, apart from those at the vertices, but as U is a real transformation, there must be just three.

The other curve of this type is

$$x^3 z^2 + z^3 y^2 + y^3 x^2 = 0; \tag{12}$$

it has a cusp of the first kind at each vertex and is invariant under U and

$$R \equiv \begin{pmatrix} x & y & z \\ x & \beta^2 y & \beta^3 z \end{pmatrix}, \quad \beta^7 = 1.$$

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* The theorem that if a point P be successively transformed into $P_1, P_2, \ldots P_{n-1}, P$ by a cyclic collineation of order n, the points P_i all lie on a conic is not necessarily true when the collineation is not real. In both statements of the theorem, a real collineation is tacitly presupposed. See J. Lüroth, "Das Imaginäre in der Geometrie und das Rechnen mit Würfen," *Math. Annalen*, vol. 11, p. 84, and "Ueber cyclisch-projectivische Punktgruppen . . . ," ibid, vol. 13, p. 304. A. Ameseder, "Theorie der cyclischen Projectivitäten," *Wiener Sitzungsberichte*, vol. 98, IIa, p. 290.

It has 21 points of inflexion of which three are real. Neither of these curves is invariant under any other collineation.

12. The most general c_5 which allows the three harmonic homologies U_1, U_2, U_3 is

$$A\sigma_1^5 + B\sigma_1^3\sigma_2 + C\sigma_1^2\sigma_3 + D\sigma_1\sigma_2^2 + E\sigma_2\sigma_3 = 0, \tag{13}$$

wherein σ_i is an elementary symmetric function of x, y, z of weight i. The centers of these homologies lie on $\sigma_1 = 0$ and their axes intersect in (1, 1, 1), pole of $\sigma_1 = 0$ as to the triangle of reference. σ_1 cuts c_5 in three points of inflexion, and two other points defined by $x^2 + xy + y^2 = 0$, which are the same for all curves of the system. The tangents at the points of inflexion will all pass through the pole if $3D + E = 0$. In this case all curves of the net have 13 points in common. The tangents at the simple intersection of σ_1, σ_5 are $x + \omega y + \omega^2 z = 0$. They intersect at (1, 1, 1) for every curve of the net.

Each conic of the pencil

$$(x + \omega y + \omega^2 z)\,(x + \omega^2 y + \omega z) - k\,(x + y + z)^2 = 0$$

is invariant in the group generated by U_1, U_2, U_3, hence if it pass through a point of inflexion it will also pass through five others. *The* 42 *points of inflexion are arranged by sizes on seven conics belonging to a double contact pencil. The other three are on the line joining the points of contact.* These points on each c_2 are in three-fold involution, having $\sigma_1 = 0$ for Pascal line. Similarly, a conic of the pencil touching a double tangent will touch five others; they form a system in triple involution, having (1, 1, 1) for Brianchon point. The 24 remaining common tangents to c_2 and c_5 form four perspective quadrilaterals.

The six common tangents of a set may be expressed by t, tU_1, tU_2, tU_3, tU_1U_2, tU_1U_3.

Of the 15 intersections of these lines, three lie on each axis, and the remaining six lie on a conic of the pencil, defining a three-fold involution.

13. An interesting particular case is furnished by the curve

$$x^5 + y^5 + z^5 = 0 \tag{14}$$

which is also invariant under V_1, V_2, V_3 making with U_1, U_2, U_3 a group of order 150.*

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* For a discussion of the G_{96} leaving a c_4 of similar equation invariant, see Dyck "Notiz über eine reguläre Riemannsche Fläche von Geschlecht drei und die zugehörige Normalcurve vierter Ordnung.' *Math. Ann.* vol. 17, (1887), pp. 510–516.

The 45 inflexions are arranged by fives on the sides of the triangle of reference, each one counting for three; the five-point-contact tangents pass through the opposite vertices. Here there are but two conics of the pencil

$$(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)-k(x+y+z)^2=0$$

which contain points of inflexion, since these two conics and the line $\sigma_1=0$ contain all 15 points.

The line joining any two points of inflexion will always pass through a third. There are 25 such lines; their equations are of the form

$$x+\theta^l y+\theta^k z=0 \qquad l, k=1,\ldots.5.$$

If we denote the point on $x=0$ in which $y+\theta^i z=0$ cuts it by x_i and similarly for y, z and finally replace x_i, y_k, z_l by (i, k, l) we may say: Three points x_i, y_k, z_l lie on a line when $i+k+l\equiv 0$ (mod. 5). The equation of the line may then be written $x+\theta^l y+\theta^k z=0$, and in five other equivalent forms. The lines (4, 4, 2), (1, 3, 1); (1, 2, 2), (4, 3, 3); (4, 2, 4), (1, 1, 3); (3, 3, 4), (2, 1, 2); (2, 4, 4), (3, 1, 1); (3, 4, 3), (2, 2, 1) intersect on $x-y$. Of these 12 lines, each intersects two others in points not at inflexions nor on an axis of homology. By U_1, the sum of the second and third symbols in any line remains constant, and symmetrically for the others. By transforming U_1, U_2, U_3 through V_1, etc., we see that G_{150} *contains* 15 *harmonic homologies.* Each of the 25 lines is invariant in three homologies whose centers are the three points of inflexion lying upon it. The three axes of homology pass through the pole of the line of centers as to the triangle of reference, from which two tangents can be drawn to c_5, meeting it at the residual intersection of (i, k, l). The pencil of conics in each case remains invariant, hence: *The fifteen points of inflexion are arranged by sixes on* 50 *conics, by threes on* 25 *lines and by fives on three lines.*

Through the nine points of inflexion lying on any three lines a pencil of c_3 can be passed, but no curve of any pencil can contain a tenth point of inflexion without becoming the sides of the triangle of reference. Similarly for the pencil formed by any line and either of inflexional conics, as

$$xyz+\lambda(x+y+z)c_2=0.$$

Similar configurations exist for the bitangents. Three are absorbed in each inflexional tangent, and five proper ones pass through each point of inflexion.

14. Of the preceding types, those which are invariant under more than one cyclic perspective are here arranged for reference.

$$axy^4 + bz^5 = 0. \quad G_{20}. \tag{1}$$

$$axy^2z^2 + b(x^5 + z^5) = 0. \quad G_{10}. \tag{2}$$

$$az^2(x^3 + ay^3) + bx^2(x^3 + by^3) = 0. \quad G_6. \tag{3}$$

$$ax^2y^3 + bz^5 = 0. \quad G_{30}. \tag{4}$$

$$x^3\phi_2(y, z) + \phi_5(y, z) = 0. \quad G_6. \tag{5}$$

$$ax^3y^2 + by^5 + cz^5 = 0. \quad G_{15}. \tag{6}$$

$$ax^3yz + b(y^5 + z^5) = 0. \quad G_{30}. \tag{7}$$

$$ax^4y + b(y^5 + z^5) = 0. \quad G_{20}. \tag{8}$$

$$x^5 + \phi_5(y, z) = 0. \quad G_{10}. \tag{9}$$

$$z^3y^2 - x(x^4 + y^4) = 0. \quad G_{12}. \tag{10}$$

$$x^4z + z^4y + y^4x = 0. \quad G_{39}. \tag{11}$$

$$x^3z^2 + z^3y^2 + y^3x^2 = 0. \quad G_{21}. \tag{12}$$

$$A\sigma_1^5 + B\sigma_1^3\sigma_2 + C\sigma_1^2\sigma_3 + D\sigma_1\sigma_2^2 + E\sigma_2\sigma_3 = 0. \quad G_6. \tag{13}$$

$$x^5 + y^5 + z^5 = 0. \quad G_{150}. \tag{14}$$

Among these types, the only ones that have more than one harmonic homology are (2), (7), (13), (14), the numbers being respectively 5, 5, 3, 15.

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